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# TWO-DIMENSIONAL TRANSPORT MODELS FOR THE LOWER LAYERS OF THE ATMOSPHERE

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**Abstract**—This paper describes two boundary value problems whose solutions assist in the interpretation of two-dimensional heat and mass transfer in the lower layers of the atmosphere.

One model considers the transient diffusion of mass into a semi-infinite two-region system; one layer is finite in thickness and the other one is infinitely thick. The diffusivities of the two layers are different. Mass is suddenly and continuously added to the boundary of the finite layer causing diffusion throughout the whole system.

A second model considers two-dimensional heat or mass transfer in a semi-infinite convection system where the boundary heat or mass flux distribution is sinusoidal with downwind distance. Uniform velocity and eddy diffusivity profiles are postulated.

The analytical results obtained for the two diffusion models are evaluated and compared to some observed phenomena for the atmosphere. Applications of the results to practical atmospheric diffusion problems in air pollution and climatology are also noted.

## NOMENCLATURE

$a$, finite layer thickness in two-region system [ft];
$c_p$, heat capacity of air [Btu/lb°F];
$C_1, C_2$, constants in equation (6);
$C_3, C_4$, constants in equation (13);
$D$, eddy diffusivity [ft²/h];
$D(z)$, height dependent eddy diffusivity [ft²/h];
$D_1$, eddy diffusivity in layer 1 [ft²/h];
$D_2$, eddy diffusivity in layer 2 [ft²/h];
$g$, Laplace transform of $G$ with respect to $t$;
$g_z$, first derivative of $g$ with respect to $z$;
$g_{zz}$, second derivative of $g$ with respect $z$;
$G$, mass concentration [lb/ft³];
$f$, function;
$n$, zero and all positive integers;
$(q/A)_0$, mean vertical heat flux in boundary layer [Btu/h ft²];
$s$, defined by $g(s) = \int_0^\infty \exp(-st)\,G(t)\,dt$ where $G(t)$ is the object function and $g(s)$ is the result function;
$t$, time [h];
$T$, potential temperature (temperature plus the product of the adiabatic lapse rate times height) [°F];

* As a student, the author first encountered Dean Boelter on the Berkeley campus of the University of California in 1941. The author attended his classes in heat transfer, mass transfer, acoustics and thermodynamics and was also one of his group of researchers who investigated aircraft heat transfer for NACA, smoke screen diffusion for the Navy, etc. In 1946, the author joined Dean Boelter's new engineering department on the Los Angeles campus to conduct research in atmospheric diffusion and boiling heat transfer and to engage in teaching activities. Dean Boelter was also the major professor for the author's doctoral program. Although the author left the University in 1950 to join the Oak Ridge National Laboratory, he kept in contact with Dean Boelter and the important, new programs being established by him at the University of California.

$T_1$, potential temperature in problem 1 [°F];
$T_2$, potential temperature in problem 2 [°F];
$T_0$, mean boundary temperature [°F];
$T'_0$, amplitude of the sinusoidal boundary temperature in the down wind direction [°F];
$u$, wind velocity [ft/h];
$u(z)$, height dependent wind velocity [ft/h];
$W$, mass flux [lb/ft$^2$ h];
$W_0$, constant mass flux at boundary of two-region system [lb/ft$^2$ h];
$x$, down wind coordinate [ft];
$x_0$, wavelength of the sinusoidal boundary temperature distribution [ft];
$z$, height above the Earth's surface [ft].

Greek symbols
$\tau$, variable of integration in the Faltung integral;
$\Gamma$, adiabatic lapse rate [°F/ft];
$\gamma$, weight density of air [lb/ft$^3$].

Moduli

$$B = \sqrt{\left(\frac{\pi u x_0}{D}\right)};$$

$$\beta^2 = \frac{(2an + 2a - z)^2}{4D_1\tau};$$

$$\lambda = \frac{\sqrt{(D_1)} - \sqrt{(D_2)}}{\sqrt{(D_1)} + \sqrt{(D_2)}};$$

$$\bar{X} = \frac{x}{x_0};$$

$$\bar{Z} = \frac{z}{x_0};$$

$$\bar{T} = \frac{T_2}{T'_0};$$

$$Q = \frac{\dfrac{\partial \bar{T}}{\partial \bar{Z}}}{\left(\dfrac{\partial \bar{T}}{\partial \bar{Z}}\right)_{\bar{Z}=0}}.$$

## 1. INTRODUCTION

A NUMBER of two-dimensional* transport solutions can be found in the literature that aid in describing diffusion systems in the lower layers of the atmosphere (see [1–4] for example).

In this paper two new, two-dimensional transport models are presented; they are believed to approximate actual atmospheric diffusion systems that are currently of practical interest. One analysis defines the transient structure of the mass transfer concentration in a two-region atmosphere in which mass is suddenly added at a boundary by means of a surface flux. A second analysis describes vertical temperature and mass concentration profiles in the atmospheric boundary layer for the condition of variable downwind boundary heat and mass flux distributions, respectively.

The derived transport solutions are evaluated for representative examples and the results compared to observed atmospheric transport phenomena and experimental diffusion data. Application of the models to problems in air pollution and micrometeorology are also discussed.

## 2. TWO-DIMENSIONAL TRANSPORT MODELS

### 2.1 *Transient mass transfer in a two-region semi-infinite atmosphere*

In 1947 and 1948, studies were conducted in Los Angeles by the University of California for the purpose of relating some of the meteorological variables to air pollution concentration [5]. The following parameters were measured within the lower few thousand feet of the atmosphere: temperature, humidity, wind speed, wind direction, and air gustiness. Temperatures and humidities were measured with the aid of radiosondes and light airplanes. Wind speeds and directions were determined by the pilot balloon technique. Smoke grenades, equipped

* In this paper, two-dimensional is taken to mean that the transport potential is a function of either one coordinate and time or of two coordinates.

with fuses, were raised and lowered in the lower layers of the atmosphere by means of a 600 $ft^3$ kite balloon. After the smoke grenades were detonated, lateral and vertical cross sections of the smoke plumes were photographed with telescopic cameras.

Air gustiness values from the smoke plume photographs were obtained as follows: One half the plume breadth was plotted as a function of distance from the smoke source. Except for a region very near the smoke source, these points fell very closely about a straight line. From the tangent of this line the ratio of a mean transverse velocity fluctuation to the average wind velocity was obtained. Typical vertical gustiness profiles in the lower 2000 ft during November and December in 1947 are shown in Fig. 1. These gustiness profiles indicate that air layers characterized by poor vertical mixing existed at an elevation of about 2000 ft and above (such conditions usually are most prevalent at that time of the year). Temperature inversions generally located in the upper layers and the unusual wind velocity structure control this stable region.

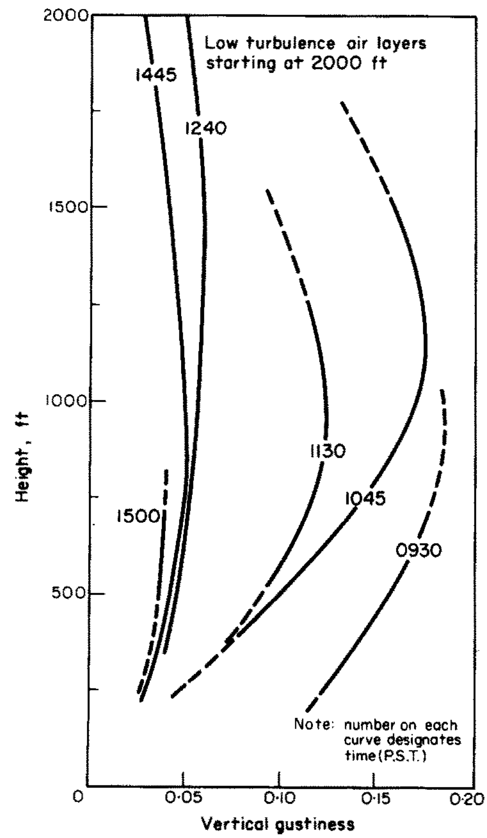


FIG. 1. Daytime vertical gustiness profiles over Los Angeles (November–December 1947).

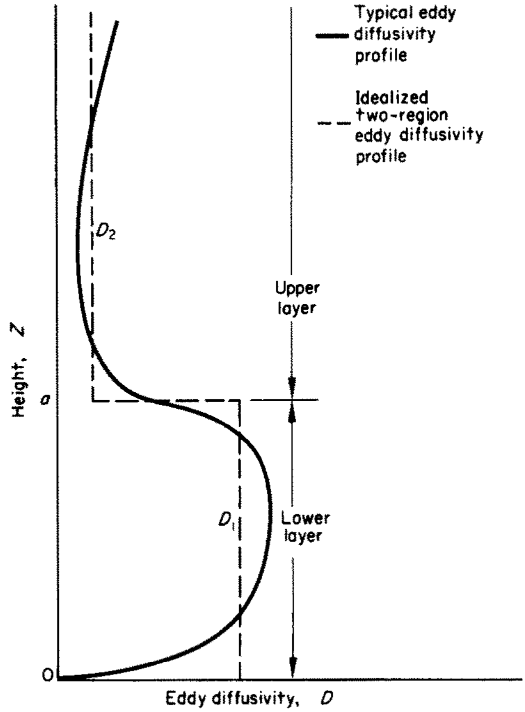


FIG. 2. Typical and idealized eddy diffusivity profiles for an atmosphere with an upper low-turbulence air layer.

It can be shown that gustiness profiles are closely related to the heat-, mass- and momentum-transfer eddy diffusivity profiles in the atmosphere [6]. Therefore, in atmospheric systems which contain upper air layers with low vertical turbulence (as occurs at times in the Los Angeles basin), an eddy diffusivity profile of the type shown in Fig. 2 is found to exist. Under such conditions, high air pollution concentrations can occur in the lower layers of the atmosphere if strong pollution flux sources are present at the surface.

In order to be able to estimate the magnitude of abnormal increases in air pollution concentration in an atmospheric system in which a low eddy diffusivity upper air layer appears (because of unusual climatological conditions), the following idealized diffusion system is proposed as a model.

2.1.1 *Idealized system.* It is postulated that at time equal to zero, mass is added at $z = 0$ at a constant rate to a semi-infinite, two-region diffusion system (see Fig. 2). The lower layer is bounded by $0 < z < a$ and the upper layer is bounded by $a < z < \infty$. The mean eddy diffusivity of the lower layer is $D_1$ and the value for the upper layer is $D_2$. Initially, the mass concentration in both layers is zero.

2.1.2 *Derivation.* The derivation of the two-region mass transport problem is only outlined here. In the region $0 < z < a$, the boundary value problem is defined by,

$$\frac{\partial G}{\partial t} = D_1 \frac{\partial^2 G}{\partial z^2} \tag{1}$$

$$G(z, 0) = 0 \tag{2}$$

$$W = W_0 = -D_1 \frac{\partial G(0, t)}{\partial z} = \text{constant}. \tag{3}$$

Equations (1–3) are the differential, initial, and boundary equations. Upon performing the Laplace transform one obtains,

$$sg(z, s) - G(z, 0) = D_1 g_{zz}(z, s) \tag{4}$$

$$-D_1 g_z(0, s) = \frac{W_0}{s}. \tag{5}$$

The solution of (4) is,

$$g = C_1 \sinh\left[z\sqrt{\left(\frac{s}{D_1}\right)}\right] + C_2 \cosh\left[z\sqrt{\left(\frac{s}{D_1}\right)}\right]. \tag{6}$$

From the boundary condition at $z = 0$ it can be shown that

$$C_1 = -\frac{W_0}{D_1}\left[\frac{1}{s}\sqrt{\left(\frac{D_1}{s}\right)}\right]. \tag{7}$$

For the region $z > a$, the boundary value problem is defined by,

$$\frac{\partial G}{\partial t} = D_2 \frac{\partial^2 G}{\partial z^2} \tag{8}$$

$$G(z, 0) = 0 \tag{9}$$

$$\lim_{z \to \infty} G(z, t) = 0. \tag{10}$$

Equations (8–10) are the differential, initial and boundary equations. Upon performing the Laplace transform one obtains

$$sg(z, s) - G(z, 0) = D_2 g_{zz}(z, s) \tag{11}$$

$$\lim_{z \to \infty} g(z, s) = 0. \tag{12}$$

The solution of (11) is

$$g = C_3 \exp\left[z\sqrt{\left(\frac{s}{D_2}\right)}\right] + C_4 \exp\left[-z\sqrt{\left(\frac{s}{D_2}\right)}\right]. \tag{13}$$

From the boundary conditions at $z = \infty$ it can be shown that $C_3 = 0$.

The constants $C_2$ and $C_4$ can be evaluated with the aid of the interface conditions

$$\left.\begin{aligned} g(a-0, s) &= g(a+0, s) \\ D_1 g_z(a-0, s) &= D_2 g_z(a+0, s). \end{aligned}\right\} \tag{14}$$

Upon substituting equations (6) and (13) into equation (14) one obtains two equations in two unknowns, $C_2$ and $C_4$. It can be shown that $C_2$ can be expressed as,

$$C_2 = \frac{\left[\frac{W_0}{s}\sqrt{\left(\frac{D_2}{D_1}\right)}\right]\sinh\left[a\sqrt{\left(\frac{s}{D_1}\right)}\right] + \frac{W_0}{s}\cosh\left[a\sqrt{\left(\frac{s}{D_1}\right)}\right]}{D_2\sqrt{\left(\frac{s}{D_2}\right)}\cosh\left[a\sqrt{\left(\frac{s}{D_1}\right)}\right] + D_1\sqrt{\left(\frac{s}{D_1}\right)}\sinh\left[a\sqrt{\left(\frac{s}{D_1}\right)}\right]}. \tag{15}$$

After substituting equation (15) into equation (6) and rearranging the expression one obtains

$$g = \frac{\frac{W_0}{s}\cosh\left[(a-z)\sqrt{\left(\frac{s}{D_1}\right)}\right] + \frac{W_0}{s}\sqrt{\left(\frac{D_2}{D_1}\right)}\sinh\left[(a-z)\sqrt{\left(\frac{s}{D_1}\right)}\right]}{D_2\sqrt{\left(\frac{s}{D_2}\right)}\cosh\left[a\sqrt{\left(\frac{s}{D_1}\right)}\right] + D_1\sqrt{\left(\frac{s}{D_1}\right)}\sinh\left[a\sqrt{\left(\frac{s}{D_1}\right)}\right]}. \tag{16}$$

The last term in equation (16) may be reduced by expressing the hyperbolic terms by exponential functions yielding,

$$\frac{W_0}{\sqrt{(D_1)}+\sqrt{(D_2)}}\sqrt{\left(\frac{D_2}{D_1}\right)}\sum_0^\infty \frac{\lambda^n}{s\sqrt{s}}\left\{\exp\left[-(2an+z)\sqrt{\left(\frac{s}{D_1}\right)}\right] - \exp\left[-(2an+2a-z)\sqrt{\left(\frac{s}{D_1}\right)}\right]\right\}. \tag{17}$$

The first term in equation (16) may be reduced in a similar manner yielding,

$$\frac{W_0}{\sqrt{(D_1)}+\sqrt{(D_2)}}\sum_0^\infty \frac{\lambda^n}{s\sqrt{s}}\left\{\exp\left[-(2an+z)\sqrt{\left(\frac{s}{D_1}\right)}\right] + \exp\left[-(2an+2a-z)\sqrt{\left(\frac{s}{D_1}\right)}\right]\right\}. \tag{18}$$

The sum of equations (17) and (18) yields the result function, $g$. In order to obtain the object function or the solution, $G$, the inverse Laplace transform must be performed for equations (17) and (18). For instance,

$$L^{-1}\left\{\frac{1}{s\sqrt{s}}\exp\left[-(2an+2a-z)\sqrt{\left(\frac{s}{D_1}\right)}\right]\right\} = L^{-1}\left[f_1(s)f_2(s)\right]$$

$$= \int_0^t 1L^{-1}\left\{\frac{-\exp\left[(2an+2a-z)\sqrt{\left(\frac{s}{D_1}\right)}\right]}{\sqrt{s}}\right\}\mathrm{d}\tau = +\frac{(2an+2a-z)}{\sqrt{D_1\pi}}\int_{\frac{2an+2a-z}{2\sqrt{(D_1t)}}}^\infty \frac{\exp(-\beta^2)}{\beta^2}\mathrm{d}\beta. \tag{19}$$

Upon integrating the above integral by parts and defining

$$2\int_{\frac{2an+2a-z}{2\sqrt{(D_1t)}}}^\infty \exp(-\beta^2)\,\mathrm{d}\beta \equiv \sqrt{\pi}\left[\text{complimentary error function of}\frac{2an+2a-z}{2\sqrt{(D_1t)}}\right].$$

Equation (19) can be expressed as

$$L^{-1}\left\{\frac{1}{s\sqrt{(s)}}\exp\left[-(2an+2a-z)\sqrt{\left(\frac{s}{D_1}\right)}\right]\right\} = +\left[2\sqrt{\left(\frac{t}{\pi}\right)}\right]\exp\left[-\frac{(2an+2a-z)^2}{4D_1t}\right] - \frac{2an+2a-z}{D_1}\operatorname{erfc}\left[\frac{(2an+2a-z)}{2\sqrt{(D_1t)}}\right].$$

The remaining inverse transform can be evaluated in the same way.
Thus the mass-transfer solution for the region $0 < z < a$ is,

$$G = \frac{W_0}{\sqrt{(D_1)} + \sqrt{(D_2)}} \sum_0^\infty \lambda^n \left\{ + 2\left[\sqrt{\left(\frac{t}{\pi}\right)}\right] \exp\left[-\frac{(2an+z)^2}{4D_1 t}\right] - \frac{(2an+z)}{\sqrt{(D_1)}} \operatorname{erfc}\left(\frac{2an+z}{2\sqrt{(D_1 t)}}\right) \right.$$
$$\left. + 2\left[\sqrt{\left(\frac{t}{\pi}\right)}\right] \exp - \frac{(2an+2a-z)^2}{4D_1 t} - \frac{(2an+2a-z)}{\sqrt{(D_1)}} \operatorname{erfc}\left[\frac{2an+2a-z}{2\sqrt{(D_1 t)}}\right] \right\}$$
$$+ \frac{W_0}{\sqrt{(D_1)} + \sqrt{(D_2)}} \frac{D_2}{D_1} \sum_0^\infty \lambda^n \left\{ + 2\left[\sqrt{\left(\frac{t}{\pi}\right)}\right] \exp\left[-\frac{(2an+z)^2}{4D_1 t}\right] - \frac{(2an+z)}{\sqrt{(D_1)}} \operatorname{erfc} \frac{2an+z}{2\sqrt{(D_1 t)}} \right.$$
$$\left. - 2\left[\sqrt{\left(\frac{t}{\pi}\right)}\right] \exp - \frac{(2an+2a-z)^2}{4D_1 t} + \frac{(2an+2a-z)}{\sqrt{(D_1)}} \operatorname{erfc}\left[\frac{2an+2a-z}{2\sqrt{(D_1 t)}}\right] \right\}. \quad (20)$$

The solution for $z > a$ may be obtained in a similar manner.

2.1.3 *Results.* The solution for the two-region, transient diffusion system, equation (20), has been applied to a representative problem. Consider a system in which $a = 2000$ ft, $D_1 = 10^6$ $ft^2/h$ and $D_2 = 10^5$ $ft^2/h$. The increase in ground level air pollution concentration with time (after the origination of the upper layer diffusion resistance) was calculated and the results presented in Fig. 3; the concentration is expressed

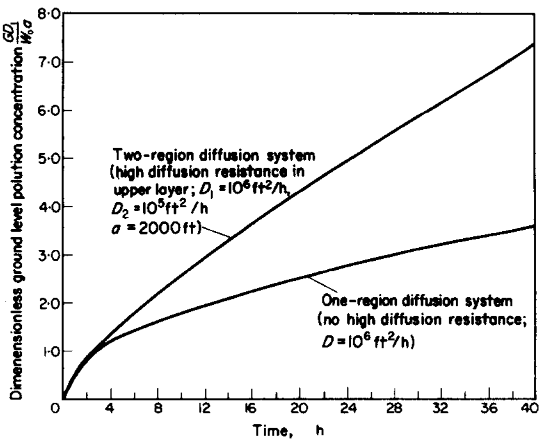


FIG. 3. Dimensionless ground level air pollution concentration variations with time for a single and two-region atmosphere.

in dimensionless form. For purposes of comparison, calculations are also shown in Fig. 3 for a second system which does not contain an upper diffusion layer resistance (i.e. a single semi-infinite layer with a diffusivity, $D = 10^6$ $ft^2/h$). Note the significant reduction in pollution concentration after a number of hours have passed when the high diffusion resistance in the upper layer is not present. Such transient air pollution concentration variations have been observed in actual diffusion systems (such as the Los Angeles basin) when abnormal reductions in turbulence levels originate in upper air layers.

### 2.2 *Boundary-layer transport for the condition where surface fluxes vary periodically with down wind distance*

One of the chief endeavors of a micrometeorologist is to measure potential gradients (temperature, humidity and velocity) and boundary fluxes (heat, mass and momentum) and then calculate the corresponding eddy diffusivities. From these results, the effects of atmospheric stability on heat-, mass- and momentum-transfer and their inter-relationships are investigated. In this regard, one of the factors that frequently prevents such turbulence or transport generalizations is the existence of non-uniform boundary potentials or fluxes. For example, how is a non-uniform earth surface heat flux or temperature distribution in the downwind direction reflected in the vertical air temperature profiles and their subsequent interpretation of atmospheric stability? A related question is, if vertical convective heat fluxes are being measured at various elevations above the ground at a given station, what variations in the measurements are to be expected under such conditions?

Experimental information has been obtained on this problem at the climatological site of the University of California at Davis [7]. Wet and dry bulb temperature profiles have been measured over a large grass field under conditions of downwind variations in temperature and humidity at the Earth's surface. Such data indicate that in nature, earth surface temperature and humidity contour plots are typified by classical peaks and valleys more or less evenly spaced. The highs and lows can be related to variations in the ground moisture content, composition, roughness, absorptivity and vegetation. If one plots the potential or flux variation at the Earth's surface in the downwind direction, a sinusoidal function results as a first approximation. For example, two specific humidity profiles were transposed from a ground humidity contour plot measured for a rye grass field at Davis, California [7] and shown in Fig. 4; corresponding temperature profiles having two degree Fahrenheit amplitudes are very similar in appearance to the humidity profiles. Larger amplitude variations occur when there are local changes in vegetation.

Only limited information on this problem can be found in the literature. Philip [4] has presented a number of boundary-layer diffusion problems together with approximate and asymptotic solutions. Step function boundary concentrations and fluxes were specified. The analysis presented here relates to a system with a periodic or sinusoidal boundary concentration or flux. The solution for this simplified two-dimensional transport model is evaluated for a specific case and utilized to predict distortions in transport potential profiles. From the solution the relative importance of the diffusivity, wind velocity and the wave length of the boundary potential or flux variation is established.

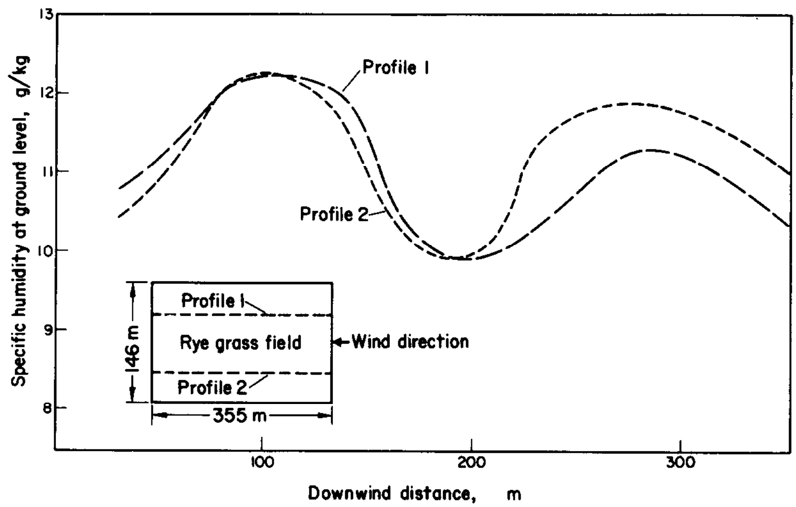


FIG. 4. Specific humidity profiles in downwind direction at ground level.

2.2.1 *Idealized system.* The idealized, two-dimensional transport model (applied below to heat transfer) is defined by the following postulates:

1. The non-uniform earth surface temperature varies sinusoidally in the direction of the wind velocity.
2. The temperature gradient at $z = \infty$ is a constant value.
3. The wind velocity profile is uniform.

4. The vertical eddy diffusivity profile is uniform.
5. Lateral eddy transport is small compared to lateral convection.
6. Steady state exists.

2.2.2 *Derivation.* The boundary condition at $z = \infty$ was postulated to be a constant temperature gradient, $\partial T/\partial z$, in order to reflect the presence of a mean vertical positive or negative heat flux (averaged over a horizontal plane). Under these conditions, a general system is defined. For example, during the day there is a net heat addition (unstable atmosphere) and at night a net heat subtraction (stable atmosphere). The intermediate case of zero net heat addition at the boundary defines a neutral atmosphere. The boundary value problem to be solved is defined by the following differential and boundary equations:

$$u\frac{\partial T}{\partial x} = D\frac{\partial^2 T}{\partial z^2} \tag{21}$$

$$T(z = 0, x) = T'_0 \cos 2\pi x/x_0 + T_0 \tag{22}$$

$$\frac{\partial T}{\partial z}(z = \infty, x) = -\frac{(q/A)_0}{\gamma c_p D}. \tag{23}$$

Equation (21) is the well-known transport relation including the idealizations described previously for this model. Equation (22) defines the sinusoidal boundary temperature distribution in the downwind direction. Equation (23) relates the temperature gradient at great distances above the boundary to the mean vertical heat flux in the boundary layer.

It can be shown that the solution of equations (21–23) is equal to the sum of the solutions of the following two boundary value problems:

Problem 1:

$$0 = \frac{D}{u}\frac{\partial^2 T_1}{\partial z^2} \tag{24}$$

$$T_1(z = 0, x) = T_0 \tag{25}$$

$$\frac{\partial T_1}{\partial z} = -\frac{(q/A)_0}{\gamma c_p D}. \tag{26}$$

Problem 2:

$$\frac{\partial T_2}{\partial x} = \frac{D}{u}\frac{\partial^2 T_2}{\partial z^2} \tag{27}$$

$$T_2(z = 0, x) = T'_0 \cos 2\pi x/x_0 \tag{28}$$

$$\frac{\partial T_2}{\partial z}(z = \infty, x) = 0. \tag{29}$$

The subscripts 1 and 2 refer to problems 1 and 2, respectively.

Problem 1 represents an atmospheric boundary layer with a uniform interface temperature or heat flux. The simple temperature solution* is,

$$T_1 = T_0 - \frac{(q/A)_0}{\gamma c_p D} z. \tag{30}$$

Problem 2 represents an atmospheric boundary layer with a sinusoidal boundary temperature distribution in the downwind direction. The mean boundary temperature is zero and the temperature gradient at great distances above the boundary is also equal to zero. Equations (27–29) are similar to the ones for the classical problem of periodic heat conduction in a slab. The solution can be obtained by the separation of variables technique. The resulting complex equation is reduced to a real function to satisfy the real periodic boundary condition at $z = 0$,

$$T_2 = T'_0 \exp\left[-B\frac{z}{x_0}\right]\cos\left(2\pi\frac{x}{x_0} - B\frac{z}{x_0}\right). \tag{31}$$

It is more convenient to express temperatures and vertical heat fluxes in dimensionless form. The dimensionless temperature equation for problem 2 is,

$$\bar{T} = \exp(-B\bar{Z})\cos(2\pi\bar{X} - B\bar{Z}). \tag{32}$$

The dimensionless vertical temperature gradient

* If the uniform eddy diffusivity is replaced by one that increases linearly with height $z$, the classical logarithmic temperature profile is obtained instead of the simplified linear profile [equation (30)].

which is proportional to the vertical convective heat flux is,

$$\frac{\partial \overline{T}}{\partial \overline{Z}} = -B\exp(-B\overline{Z})[\cos(2\pi\overline{X} - B\overline{Z}) - \sin(2\pi\overline{X} - B\overline{Z})]. \quad (33)$$

The temperature solution to the boundary value problem defined by equations (21–23) is the sum of the temperature solutions given by equations (30) and (31). Thus temperature and vertical heat flux profiles for an atmospheric boundary layer with a sinusoidal boundary temperature distribution in the down wind direction are defined.

vertical temperature gradient, $\partial\overline{T}/\partial\overline{Z}$. The contour plumes depict the vertical temperature and heat flux variability as a function of downwind position. A graph of the ratio of the vertical heat flux at a given elevation to the heat flux at the boundary (denoted as $Q$) can be seen in Fig. 7. Note the large variations in vertical heat flux that can be encountered in a neutral boundary layer with a periodic earth surface temperature distribution (the extreme case). Such variations become significantly smaller, of course, for stable and unstable atmospheres (where mean negative or positive heat fluxes exist in the boundary layer). These results serve to illustrate the errors that can be made by using ground

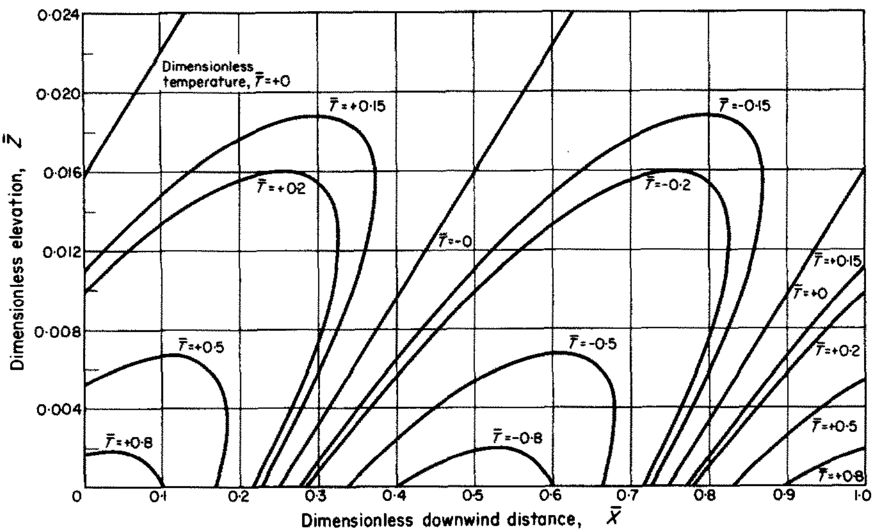


FIG. 5. Dimensionless temperature contours (neutral case, $B = 100$).

2.2.3 *Results.* An illustrative example depicting temperature and vertical heat flux profiles in a boundary layer with a sinusoidal boundary temperature distribution has been evaluated for neutral atmospheric stability $[(q/A)_0 = 0]$ for the specific condition, $B = 100$. Figure 5 presents profiles of the dimensionless temperature, $\overline{T}$, as a function of the dimensionless vertical and downwind positions. Similarly, Fig. 6 shows profiles of the dimensionless level vertical convective flux measurements in the place of those existing at other elevations above the ground where diffusivities are being evaluated from fluxes and potential gradients. The dimensionless temperature profiles as a function of downwind position are shown in Fig. 8 for this neutral case. The decay rates (with respect to elevation above the ground) of the temperature and heat flow amplitudes are controlled by the dimensionless modulus $B$ which

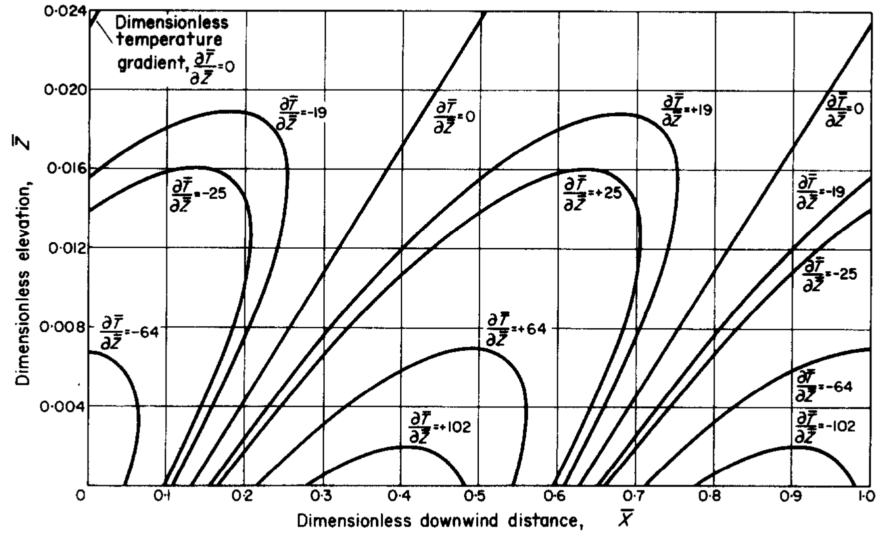


FIG. 6. Dimensionless vertical temperature gradient contours (neutral case, $B = 100$).

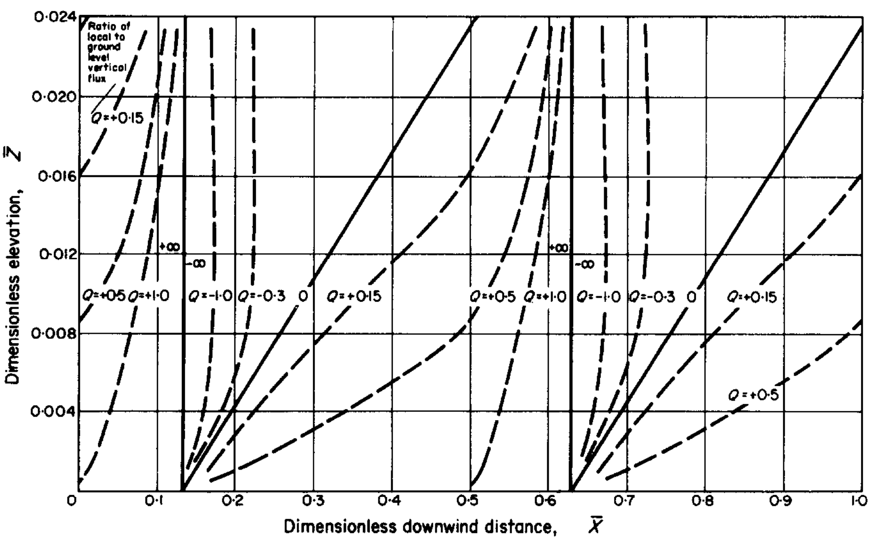


FIG. 7. Dimensionless vertical heat flux contours (neutral case, $B = 100$).

is a function of the boundary temperature wave length $x_0$ and the mean wind speed and eddy diffusivity.

Some of the University of California vertical temperature profile data referenced previously for variable downwind surface heat flux conditions were examined on the basis of the two-dimensional diffusion model presented above. A mean experimental ratio of the temperature amplitude at the 310 cm elevation to the value at the ground was 0·12 for a mean wind velocity of 7·0 m/s, a ground temperature wave length of about 200 m and a mean eddy diffusivity of about 1800 cm²/s. The predicted temperature

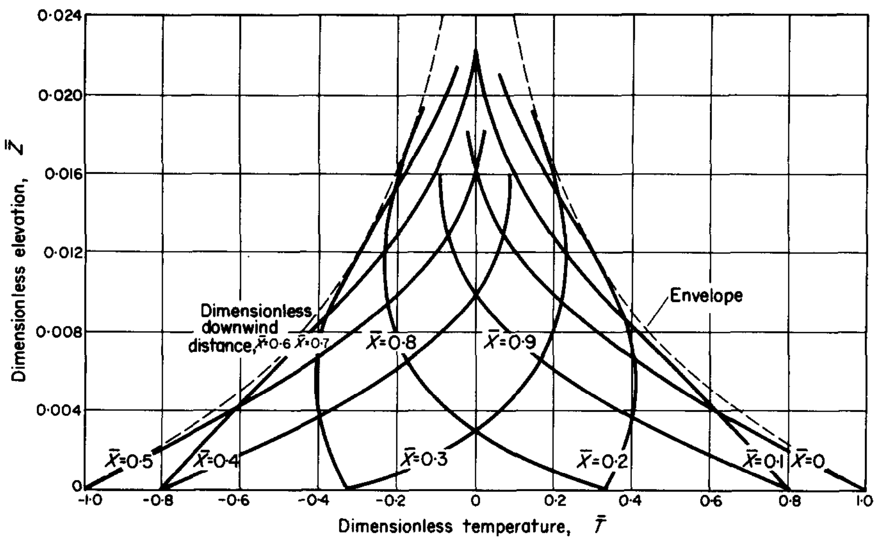


FIG. 8. Dimensionless temperature profiles as a function of $\bar{X}$ (neutral case, $B = 100$).

amplitude reduction, $\exp(-B\bar{Z})$, from equation (31) was found to be 0·093 in comparison to the experimental value of 0·12. This agreement is considered to be better than expected because the actual earth surface temperature variation with downwind position was only approximately sinusoidal in character.

### 3. DISCUSSION

The first boundary value problem discussed in this paper relates to transient diffusion in a two-region system where mass is added at a uniform rate at one boundary. This model is considered to represent a first approximation to an important air pollution system of the type that can be found in the Los Angeles Basin (specifically where a high diffusion resistance develops in the upper air layers and subsequently controls the ground level pollution concentrations). It is recognized, however, that the actual diffusion system can be more complicated than the idealized one presented in that time variations in the boundary mass flux and the eddy diffusivity profiles can complicate the problem. A number of extensions of this model to include such variations have been made but not shown here. For example, if the pollution boundary sources vary from day to night, a more general solution can be obtained by superposing the present solution using different boundary mass fluxes. Time variable diffusivity problems can also be treated on a periodic [3] or step function basis.

It is believed that the results of the second boundary value problem discussed in this paper will help the micrometeorologist in making better measurements as well as in the interpretation of the data. For example, if it is desired to deduce local thermal turbulence structure at some elevation in the boundary layer, both local temperature gradients and heat fluxes (rather than boundary fluxes) must be used if the boundary fluxes and temperatures are non-uniform. Also, from the theory it is possible to (1) calculate local flux variations using experimental measurements for ground temperature or flux variations or (2) choose a site where potential or flux variations at the ground are a minimum (using the theory to set the permissible error). The second boundary value problem considered includes the idealization that the wind velocity and diffusivity profiles are uniform. In the lower layers of the atmosphere, both the wind velocity and eddy diffusivity actually increase with height; the velocity distributions are normally represented by logarithmic expressions and the diffusivities are usually linear or exponential functions [8]. Consequently, the general diffusion equation has been considered, namely,

$$u(z)\frac{\partial T}{\partial x} = D(z)\frac{\partial^2 T}{\partial z^2} + \frac{\partial D(z)}{\partial z}\frac{\partial T}{\partial z}. \quad (34)$$

Linear and exponential functions of $u(z)$ and $D(z)$ have been substituted into equation (34) and solutions obtained in terms of height dependent functions more complicated than exponentials (such as Hankel functions). Such solutions more accurately define energy transport in the atmospheric boundary layer having sinusoidal boundary heat or mass fluxes. The resulting two-dimensional temperature profiles are distorted versions of those elementary ones shown in Fig. 5, reflecting the height dependent velocity and eddy diffusivity functions.

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**Résumé**—Cet article décrit deux problèmes de valeurs aux limites dont les solutions aident à l'interprétation du transport bidimensionnel de chaleur et de messe dans les couches les plus basses de l'atmosphère.

Un modèle considère la diffusion de masse en régime transitoire dans un système semi-infini avec deux domaines; l'une des couches a une épaisseur finie et l'autre est infiniment épaisse. Les diffusivités des deux couches sont différentes. La masse est ajoutée brutalement et continuellement à la frontière de la couche finie provoquant la diffusion à travers tout le système.

Un deuxième modèle considère le transport de chaleur ou de masse dans un système convectif semi-infini où la distribution de flux de chaleur ou de masse à la frontière varie sinusoïdalement avec la distance dans la direction du vent.

On a supposé que les profils de vitesse et de diffusivité turbulente étaient uniformes. Les résultats théoriques obtenus pour les deux modèles de diffusion sont évalués et comparés à quelques phénomènes observés dans l'atmosphère. Des applications des résultats à des problèmes pratiques de diffusion atmosphérique dans la pollution atmosphérique et la climatologie ont également été relevées.



**Zusammenfassung**—Die Arbeit beschreibt zwei Grenzwertprobleme, deren Lösungen die Interpretation des zweidimensionalen Wärme- und Stoffübergangs in den unteren Schichten der Atmosphäre erleichtert.

Ein Modell behandelt den instationären Stofftransport in ein halbunendliches Zweibereichssystem; ein Schicht besitzt endliche Dicke und die andere ist unendlich dick. Die Durchlässigkeiten der beiden Schichten sind unterschiedlich. Der Begrenzung der endlichen Schicht wird plötzlich und kontinuierlich Masse zugeführt, wodurch Diffusion im gesamten System auftritt.

Ein zweites Modell behandelt den zweidimensionalen Wärme- und Stoffübergang in einem halbunendlichen Konvektionssystem, wobei die Grenzwärme- und Stoffstromdichte stromabwärts sinusförmige Verteilung aufweisen. Gleichmässige Profile für Geschwindigkeits- und Wirbelverteilung sind vorausgesetzt.

Die für die beiden Diffusionsmodelle erhaltenen analytischen Ergebnisse wurden ausgewertet und mit einigen in der Atmosphäre beobachteten Erscheinungen verglichen. Anwendungen der Ergebnisse auf praktische atmosphärische Diffusionsprobleme wie bei der Luftverunreinigung und der Wetterkunde sind enbenfalls mitgeteilt.

**Аннотация**—Приводится решение двух краевых задач двумерного тепло-и массобмена в нижних слоях атмосферы. Одна модель рассматривает нестационарную диффузию массы в полуограниченной системе, состоящей из двух областей; один слой имеет конечную толщину, а другой бесконечную. Коэффициенты диффузии слоев различны. Масса мгновенно и непрерывно поступает на границу бесконечного слоя, вызывая диффузию во всей системе.

Во второй модели рассматривается тепло-и массообмен в полуограниченной конвективной системе с синусоидальным законом распределения потоков тепла и массы вдоль границы. Скорость и коэффициент турбулентного перемешивания принимаются однородными.

Полученные тероетические результаты для этих моделей диффузии оценены и сравнены с некоторыми явлениями, наблюдаемыми в атмосфере. Также отмечена возможность применения результатов для решения практических задач загрязнения воздуха и климатологии.